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Quantum Theory of Measurement and the Polar Decomposition of an Interaction

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The measurement-theoretic content of the polar decomposition of an interaction is analyzed. It is shown that the polar decomposition arises exactly from the strong correlation premeasurement of a discrete physical quantity.

1. INTRODUCTION

Kochen (1985) developed a new interpretation of quantum mechanics called the witnessing interpretation. In Kochen (1988*a*) this interpretation is further developed under the name of the perspective interpretation of quantum mechanics. This interpretation is formally based on the so-called polar decomposition of an interaction. In this note I shall identify the polar decomposition within the usual formulation of quantum measurement theory, as given, e.g., in Beltrametti and Cassinelli (1981) or Beltrametti *et al.* (1990). I show here that the polar decomposition arises exactly from the so-called strong correlation premeasurements of a discrete quantity. This formal bridge between the polar decomposition and the usual measurement theory points up the difference between the usual Born interpretation of quantum mechanics and the witnessing interpretation. Moreover, this bridge reveals the dangers in any uncritical application of the witnessing interpretation, and it points out the need for a systematic development of this interpretation.

2. MATHEMATICAL PRELIMINARIES. THE POLAR DECOMPOSITION

Let **H** be a complex separable Hilbert space, with the inner product $\langle\cdot|\cdot\rangle$. Let **L(H)** denote the set of bounded linear operators on **H** (equipped

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with the usual algebraic and topological structures). The set of all trace-class operators on $\mathbf{H}$ is denoted as $\mathbf{T}(\mathbf{H})$, and recall that it is a Banach space with respect to the trace norm $\|\cdot\|_1$. The set $\mathbf{T}(\mathbf{H})_1^+$ consists of the positive (i.e., $T \geq 0$) trace-one (i.e., $\|T\|_1 = \mathrm{tr}[T] = 1$) operators on $\mathbf{H}$.

Let $\mathbf{H}_a$ be another complex separable Hilbert space. The (Hilbert space) tensor product of $\mathbf{H}$ and $\mathbf{H}_a$ is denoted as $\mathbf{H} \otimes \mathbf{H}_a$. The partial trace over $\mathbf{H}_a$, say, is the positive linear map $\pi_a : \mathbf{T}(\mathbf{H} \otimes \mathbf{H}_a) \to \mathbf{T}(\mathbf{H})$ defined as follows:

$$\mathrm{tr}[\pi_a(\hat{T})A] = \mathrm{tr}[\hat{T} \cdot A \otimes I_a] \tag{1}$$

where $\hat{T} \in \mathbf{T}(\mathbf{H} \otimes \mathbf{H}_a)$, $A \in \mathbf{L}(\mathbf{H})$, and I_a is the identity operator on $\mathbf{H}_a$. Similarly, we have the partial trace over $\mathbf{H}$, and denote it as π. Note also that if $\{\varphi_k : k \in \mathbb{K}\} \subset \mathbf{H}$ and $\{\psi_\kappa : \kappa \in \mathbb{K}_a\} \subset \mathbf{H}_a$ are orthonormal bases, then, e.g., $\pi_a(\hat{T})$ can be expressed as

$$\pi_a(\hat{T}) = \sum_{k,\kappa,l} \langle \varphi_k \otimes \psi_\kappa | \hat{T}\varphi_l \otimes \psi_\kappa \rangle |\varphi_k\rangle\langle\varphi_l| \tag{2}$$

as $\{\varphi_k \otimes \psi_\kappa : (k, \kappa) \in \mathbb{K} \times \mathbb{K}_a\}$ is an orthonormal basis of $\mathbf{H} \otimes \mathbf{H}_a$. Here, e.g., $|\varphi_k\rangle\langle\varphi_l|$ is the bounded linear operator on $\mathbf{H}$ defined as $|\varphi_k\rangle\langle\varphi_l|(\varphi) = \langle\varphi_l|\varphi\rangle\varphi_k$, $\varphi \in \mathbf{H}$.

Let $\hat{\Phi}$ be a unit vector of $\mathbf{H} \otimes \mathbf{H}_a$, and let $|\hat{\Phi}\rangle\langle\hat{\Phi}|$, or $P[\hat{\Phi}]$, denote the one-dimensional projection operator on $\mathbf{H} \otimes \mathbf{H}_a$ defined by $\hat{\Phi}$. Clearly, $P[\hat{\Phi}] \in \mathbf{T}(\mathbf{H} \otimes \mathbf{H}_a)_1^+$, i.e., $P[\hat{\Phi}] \geq 0$ and $\mathrm{tr}[P[\hat{\Phi}]] = 1$. The "reduced states" $\pi_a(P[\hat{\Phi}])$ and $\pi(P[\hat{\Phi}])$ are also positive trace-one operators on $\mathbf{H}$ and $\mathbf{H}_a$, respectively. The vector $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ may, of course, be expressed as

$$\hat{\Phi} = \sum_{k,\kappa} c_{k\kappa}\varphi_k \otimes \psi_\kappa \tag{3}$$

in terms of the above basis $\{\varphi_k \otimes \psi_\kappa : (k, \kappa) \in \mathbb{K} \times \mathbb{K}_a\}$. (Recall also that $\sum |c_{k\kappa}|^2 = \|\hat{\Phi}\| = 1$, where $c_{k\kappa} = \langle \varphi_k \otimes \psi_\kappa | \hat{\Phi}\rangle$.) When applied together with (2), the formula (3) leads to the following expression for $\pi_a(P[\hat{\Phi}])$:

$$\begin{aligned}\pi_a(P[\hat{\Phi}]) &= \sum_{k,\kappa,l} c_{k\kappa}\bar{c}_{l\kappa} |\varphi_k\rangle\langle\varphi_l| \\ &= \sum_{k,\kappa} |c_{k\kappa}|^2 |\varphi_k\rangle\langle\varphi_k| + \sum_{\substack{k,\kappa,l \\ (k \neq l)}} c_{k\kappa}\bar{c}_{l\kappa} |\varphi_k\rangle\langle\varphi_l|\end{aligned} \tag{4}$$

Similarly, we obtain

$$\begin{aligned}\pi(P[\hat{\Phi}]) &= \sum_{k,\kappa,\lambda} c_{k\kappa}\bar{c}_{k\lambda} |\psi_\kappa\rangle\langle\psi_\lambda| \\ &= \sum_{k,\kappa} |c_{k\kappa}|^2 |\psi_\kappa\rangle\langle\psi_\kappa| + \sum_{\substack{k,\kappa,\lambda \\ (\kappa \neq \lambda)}} c_{k\kappa}\bar{c}_{k\lambda} |\psi_k\rangle\langle\psi_\lambda|\end{aligned} \tag{5}$$

The representations (4) and (5) are the natural decompositions of the "reduced states" $\pi_a(P[\hat{\Phi}])$ and $\pi(P[\hat{\Phi}])$ into the rank-one operators $|\varphi_k\rangle\langle\varphi_l|$

and $|\psi_\kappa\rangle\langle\psi_\lambda|$ associated with the chosen orthonormal basis $\{\varphi_k \otimes \psi_\kappa : (k, \kappa) \in \mathbb{K} \times \mathbb{K}_a\} \subset \mathbf{H} \otimes \mathbf{H}_a$. Clearly, these decompositions are not the canonical (i.e., spectral) decompositions of $\pi_a(P[\hat{\Phi}])$ and of $\pi(P[\hat{\Phi}])$ unless the "interference terms" in the second summands of (4) and (5) are zero.

The polar decomposition of $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ serves to point out an orthonormal basis of $\mathbf{H} \otimes \mathbf{H}_a$ such that when $\hat{\Phi}$ is expressed in terms of this basis [as in (3)], then the natural decompositions of the "reduced states" $\pi_a(P[\hat{\Phi}])$ and $\pi(P[\hat{\Phi}])$ [as in (4) and (5)] are exactly the canonical ones, so that, in particular, no "interference terms" appear in these decompositions.

Though the polar decomposition of $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ was already worked out in Kochen (1985), I sketch here the main steps to obtain it. It starts with identifying $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ with a bounded linear map $F(\hat{\Phi})\colon \mathbf{H}_a \to \mathbf{H}$ given by

$$F(\hat{\Phi}) = F\left(\sum c_{k\kappa} \varphi_k \otimes \psi_\kappa\right) = \sum c_{k\kappa} F(\varphi_k \otimes \psi_\kappa) = \sum c_{k\kappa} |\varphi_k\rangle\langle\psi_\kappa|$$

so that for any $\psi \in \mathbf{H}_a$, $F(\hat{\Phi})(\psi) = \sum c_{k\kappa} \langle\psi_\kappa|\psi\rangle \varphi_k$. Then one applies the polar decomposition theorem to the map $F(\hat{\Phi})$ to obtain a unique decomposition $F(\hat{\Phi}) = U \circ V$, where $V\colon \mathbf{H}_a \to \mathbf{H}_a$ is a positive operator and $U\colon \mathbf{H}_a \to \mathbf{H}$ is a partial isometry [i.e., $\|U\psi\| = \|\psi\|$ for any $\psi \in \operatorname{kern}(U)^\perp$] such that $\operatorname{kern}(U)^\perp = \overline{\operatorname{ran}(V)}$ [see, e.g., Reed and Simon (1972), Theorem VI.10, p. 197]. The spectral decomposition of V can now be determined. Indeed, from $\|F(\hat{\Phi})(\psi)\|^2 = \|U(V\psi)\|^2 = \|V\psi\|^2$, $\psi \in \mathbf{H}_a$, one shows that V^2 is compact. Then, using the spectral decomposition of V^2, and by the positivity of V, $V = (V^2)^{1/2}$, one obtains

$$V = \sum v_i P_i \tag{6}$$

where $v_i > 0$ for any $i = 1, 2, \ldots, N$ (N being the number of distinct eigenvalues of V, $N \in \mathbb{N}$, or $N = \infty$), $v_i \neq v_j$ for any $i \neq j$, and $P_1, \ldots, P_N$ are mutually orthogonal projection operators on $\mathbf{H}_\alpha$. [Recall that the positive numbers v_i in (6) are indeed eigenvalues of V, and $\lim v_i = 0$, if $N = \infty$. In that case 0 is the only possible accumulation point of the spectrum of V.] As $\hat{\Phi}$ is a unit vector, it also follows that $\sum v_i^2 = 1$.

Now, let $\{\gamma_{ij} : j = 1, \ldots, n(i)\}$ be an orthonormal basis of the eigenspace $P_i(\mathbf{H}_a)$, where $n(i) = \dim(P_i(\mathbf{H}_a))$. Then $\{\gamma_{ij} : i = 1, \ldots, N, j = 1, \ldots, n(i)\}$ is an orthonormal system in $\mathbf{H}_a$, and

$$V = \sum_{i=1}^{N} v_i \sum_{j=1}^{n(i)} P[\gamma_{ij}] \tag{7}$$

[Recall that if $\{\gamma_{ij}\} \subset \mathbf{H}_a$ is not an orthonormal basis, it can always be extended to one. The sum in (7) can then be extended to range over this basis with the cost that some v_i may then be zero.]

As $U: \mathbf{H}_a \to \mathbf{H}$ is a partial isometry with $\|UV\psi\| = \|V\psi\|$ for any $\psi \in \mathbf{H}_a$, the set $\{\xi_{ij}: i = 1, \ldots, N, j = 1, \ldots, n(i)\}$, with $\xi_{ij} = U\gamma_{ij}$, is an orthonormal system in $\mathbf{H}$. But then for any $\psi \in \mathbf{H}_a$,

$$F(\hat{\Phi})(\psi) = U(V\psi) = \sum v_i \sum U(P[\gamma_{ij}]\psi) = \sum v_i \sum \langle \gamma_{ij} | \psi \rangle \xi_{ij}$$

which shows that the vector $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ can now be expressed as

$$\hat{\Phi} = \sum_{i=1}^{N} \sum_{i=1}^{n(i)} v_i \xi_{ij} \otimes \gamma_{ij} \tag{8}$$

This is the polar decomposition of $\hat{\Phi}$. It is unique exactly when all the eigenvalues v_i of V are nondegenerate. In such a case we simply have

$$\hat{\Phi} = \sum v_i \xi_i \otimes \gamma_i \tag{9}$$

with $P_i = P[\gamma_i]$ for any $i = 1, \ldots, N$.

Note. Instead of identifying $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ with $F(\hat{\Phi}): \mathbf{H}_a \to \mathbf{H}$, we could equally well identify $\hat{\Phi}$ with $F'(\hat{\Phi}): \mathbf{H} \to \mathbf{H}_a$, where $F'(\varphi_k \otimes \psi_\kappa) = |\psi_\kappa\rangle\langle\varphi_k|$. Applying then the polar decomposition theorem to $F'(\hat{\Phi})$, we would get exactly the same expression (8) for $\hat{\Phi}$. [See Kochen (1985).] It is to be stressed that the polar decomposition is known in quantum mechanics at least since the work of von Neumann (1955), and it is also known as the normal or the biorthogonal decomposition.

From now on, only in order to simplify the notations, I shall consider exclusively $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ which have nondegenerate (and thus unique) polar decompositions (9). This simplification has no implications for the results discussed here.

Assume, then, that $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ has the polar decomposition (9). The "reduced states" of $\hat{\Phi}$ obtain as their natural decompositions

$$\pi_a(P[\hat{\Phi}]) = \sum v_i^2 P[\xi_i] \tag{10}$$

$$\pi(P[\hat{\Phi}]) = \sum v_i^2 P[\gamma_i] \tag{11}$$

They are simply the canonical decompositions of $\pi_a(P[\hat{\Phi}])$ and $\pi(P[\hat{\Phi}])$. To conclude, the polar decomposition of a vector $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ singles out an orthonormal system (which can be extended to an orthonormal basis) of $\mathbf{H} \otimes \mathbf{H}_a$ such that $\hat{\Phi}$ obtains the simple form (9) and the natural decompositions of the "reduced states" of $P[\hat{\Phi}]$ are the canonical ones.

Let us now turn to the quantum theory of measurement to see how the polar decomposition (9) of a (nondegenerate) vector $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$ arises there.

3. QUANTUM THEORY OF MEASUREMENT AND THE POLAR DECOMPOSITION

In the usual Hilbert space formulation of quantum mechanics the description of a physical system $\mathbf{S}$ is based on a complex separable Hilbert space **H**. Any physical quantity of **S** is represented as (and identified with) a self-adjoint operator A in **H**. Any state of the system is represented as (and identified with) an element T of $\mathbf{T}(\mathbf{H})_1^+$ of positive trace-one operators on **H**. In this representation the pure states of **S** appear as the one-dimensional projection operators $P[\varphi]$, $\varphi \in \mathbf{H}$, $\|\varphi\| = 1$. Occasionally one refers also to the unit vectors of **H** as the vector states of **S**. Let $\mathbf{B}(\mathbb{R})$ denote the family of Borel subsets of the real line $\mathbb{R}$, and let $E^A: \mathbf{B}(\mathbb{R}) \to \mathbf{L}(\mathbf{H})$ be the spectral measure of the self-adjoint operator A. Any pair (A, T) consisting of a physical quantity A and a state T defines a probability measure $E_T^A: \mathbf{B}(\mathbb{R}) \to [0, 1]$, $X \mapsto E_T^A(X) := \mathrm{tr}[TE^A(X)]$, where $[0, 1]$ is the unit interval of the real line $\mathbb{R}$. According to the *Born interpretation*, the number $E_T^A(X)$ is the probability that a measurement of A on **S** in the state T yields a result in X. [For further details of this formulation of quantum mechanics see, e.g., Beltrametti and Cassinelli (1981).]

The quantum theory of measurement is the part of quantum mechanics which investigates the measurement possibilities of quantum mechanics. It also investigates the consistency of the Born interpretation of the basic probabilities $E_T^A(X)$ of the theory.

Following the usual formulation of the quantum theory of measurement as a part of the theory of compound systems in quantum mechanics, one is led to the following notion of a premeasurement of a quantity A of the (object) system **S**. Let $\mathbf{H}_a$ be a complex separable Hilbert space (associated with the measuring apparatus **A**); let A_a be a self-adjoint operator in $\mathbf{H}_a$ (representing the so-called pointer observable of **A**). Let ψ be a unit vector of $\mathbf{H}_a$ (representing the initial vector state of **A**), and let $U: \mathbf{H} \otimes \mathbf{H}_a \to \mathbf{H} \otimes \mathbf{H}_a$ be a unitary operator (modeling the interaction between **S** and **A**). Then the 4-tuple $\langle \mathbf{H}_a, A_a, \psi, U \rangle$ is a *premeasurement* of A on **S** if it reproduces the statistics $(A, P[\varphi])$, $\varphi \in \mathbf{H}$, $\|\varphi\| = 1$, through $(A_a, \pi(P[U(\varphi \otimes \psi)]))$, i.e.,

$$\mathrm{tr}(P[\varphi]E^A(X)) = \mathrm{tr}(\pi(P[U(\varphi \otimes \psi)])E^{A_a}(X)) \tag{12}$$

for any $X \in \mathbf{B}(\mathbb{R})$.

Note. I consider here only the so-called normal premeasurements of A, i.e., those 4-tuples where the initial state of the measuring apparatus is a vector state and where the interaction is given by a unitary mapping. These assumptions imply, however, almost no loss in generality [see Busch and Lahti (1990) and references therein]. Also, the assumption that the initial state of the object system is a pure one is no restriction (Beltrametti *et al.*, 1990). Finally, note that it is not necessary to have the same value sets X on both sides of (12). In fact, it is enough that the A value sets and the A_a value sets are in one-to-one correspondence.

The "probability reproducibility condition" (12) expresses simply the idea that the probability of A taking a given value in the initial state of **S** should be the same as the probability of the pointer observable A_a assuming the same (i.e., corresponding) value in the final state of the measuring apparatus.

In order for a premeasurement of A to be a *measurement* of A we should be able to say, at the end, that the measurement led to a "definite result." To obtain this, some further conditions on the interaction as well as on the measuring apparatus must be required. This part of the measurement problem is usually called the *objectification problem*, and I shall address that later.

To obtain a characterization of the polar decomposition of a (nondegenerate) vector state $\hat{\Phi} \in \mathbf{H} \otimes \mathbf{H}_a$, $\|\hat{\Phi}\| = 1$, within the quantum measurement theory, it is sufficient to consider a class of premeasurements of the so-called simple quantities, i.e., discrete quantities with nondegenerate eigenvalues.

Let $A = \sum_{i=1}^{N} a_i P[\varphi_i]$ be a simple quantity of the object system $\mathbf{S}$. Here N is the number of distinct eigenvalues of A. The number N may be finite or infinite. Let $\mathbf{H}_a$ be a complex separable Hilbert space with the vector space dimension equal to N. Let $\{\gamma_i : i = 1, \ldots, N\}$ be an orthonormal basis of $\mathbf{H}_a$, and define $A_a = \sum_{i=1}^{N} a_i P[\gamma_i]$. Let ψ be a unit vector of $\mathbf{H}_a$. The unitary mappings $U: \mathbf{H} \otimes \mathbf{H}_a \to \mathbf{H} \otimes \mathbf{H}_a$ for which $\langle \mathbf{H}_a, A_a, \psi, U \rangle$ are premeasurements of A can then be characterized. This is done in Beltrametti *et al.* (1990) (where, in fact the problem is analyzed with a greater generality). Here I note only that $\langle \mathbf{H}_a, A_a, \psi, U \rangle$ is a premeasurement of A if and only if U is of the form

$$U(\varphi \otimes \psi) = \sum c_i \xi_i \otimes \gamma_i \tag{13}$$

where $c_i = \langle \varphi_i | \varphi \rangle$, $\varphi \in \mathbf{H}$, $\|\varphi\| = 1$, and $\{\xi_i : i = 1, \ldots, N\}$ is any (fixed) collection of unit vectors on $\mathbf{H}$ (Theorem 3.2 in Beltrametti *et al.* (1990).

The unit vector $U(\varphi \otimes \psi) \in \mathbf{H} \otimes \mathbf{H}_a$ is the final vector state of $\mathbf{S} + \mathbf{A}$ when φ and ψ are the initial vector states of $\mathbf{S}$ and $\mathbf{A}$, respectively. The final states of $\mathbf{S}$ and $\mathbf{A}$ are again given as the "reductions" of $U(\varphi \otimes \psi)$:

$$\pi_a(P[U(\varphi \otimes \psi)]) = \sum |c_i|^2 P[\xi_i] \tag{14}$$

$$\pi(P[U(\varphi \otimes \psi)]) = \sum c_i \bar{c}_j \langle \xi_i | \xi_j \rangle |\gamma_i\rangle\langle\gamma_j| \tag{15}$$

Clearly, (13) need not be, in general, the polar decomposition of the vector $U(\varphi \otimes \psi)$. I emphasize that the basic requirement on the premeasurement, namely condition (12), does not imply any orthogonality relations on the set $\{\xi_i : i = 1, \ldots, N\}$. In spite of this, (13)–(15) are the natural representations or decompositions of $\hat{\Phi} = U(\varphi \otimes \psi)$, $\pi_a(P[\hat{\Phi}])$, and $\pi(P[\hat{\Phi}])$ with respect to the interaction U.

The pure states $\{P[\xi_i]: i = 1, \ldots, N\}$ and $\{P[\gamma_i]: i = 1, \ldots, N\}$ singled out by the premeasurement $\langle \mathbf{H}_a, A_a, \psi, U \rangle$ are the natural candidates for the final pure states of $\mathbf{S}$ and $\mathbf{A}$, respectively. In fact, it is the task of the objectification problem to justify this. As a step toward a solution of the objectification problem, consider the premeasurements $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ of A, with varying U, which lead to strong correlations (1) between the possible values of A and A_a, and (2) between the distinguished pure states $P[\xi_i]$ and $P[\gamma_i]$, $i=1,\ldots,N$.

A premeasurement $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ of A has *the strong correlation property with respect to states* if

$$\rho(P[\xi_i], P[\gamma_i], U(\varphi\otimes\psi))=1 \tag{16}$$

for all $i=1,\ldots,N$, and for any initial vector state φ of $\mathbf{S}$ [for which the condition (16) is meaningful, i.e., $0\neq |c_i|^2=\langle\varphi|E^A(\{a_i\})\varphi\rangle\neq 1$]. A necessary and sufficient condition that $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ has this property is that the set $\{\xi_i: i=1,\ldots,N\}$ is orthonormal. [Theorem 4.1 in Beltrametti *et al.* (1990)]. But this then means that

$$U(\varphi\otimes\psi)=\sum |c_i|\xi_i\otimes\gamma_i \tag{17}$$

where $\{\xi_i: i=1,\ldots,N\}$ is an orthonormal set, and where the phase factor of $c_i=|c_i|e^{i\theta_i}$ has (for convenience) been incorporated into the vector ξ_i. The "reduced states" are then

$$\pi_a(P[U(\varphi\otimes\psi)])=\sum |c_i|^2P[\xi_i] \tag{18}$$

$$\pi(P[U(\varphi\otimes\psi)])=\sum |c_i|^2P[\gamma_i] \tag{19}$$

As the polar decomposition of a nondegenerate vector $\hat{\Phi}=U(\varphi\otimes\psi)$ is unique (modulo the phase factors), one can then conclude that a premeasurement $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ of A which leads to the strong correlations (16) gives always, i.e., for any $\varphi\in\mathbf{H}$, $\|\varphi\|=1$, the polar decomposition of the final state $U(\varphi\otimes\psi)$ of the compound system $\mathbf{S}+\mathbf{A}$, and thus also the canonical decompositions of the "reduced states" $\pi_a(P[U(\varphi\otimes\psi)])$ and $\pi(P[U(\varphi\otimes\psi)])$ of the component systems $\mathbf{S}$ and $\mathbf{A}$.

To complete the analysis, I next demonstrate that the polar decomposition (9) of a nondegenerate unit vector $\hat{\Phi}\in\mathbf{H}\otimes\mathbf{H}_a$ always admits the above kind of a measurement-theoretic interpretation. Indeed, let $\hat{\Phi}=\sum v_i\xi_i\otimes\gamma_i$ be the polar decomposition of $\hat{\Phi}\in\mathbf{H}\otimes\mathbf{H}_a$. Extending the orthonormal systems $\{\xi_i\}$ and $\{\gamma_i\}$ into orthonormal bases, we have the simple quantities $A=\sum a_iP[\xi_i]$ and $A_a=\sum a_iP[\gamma_i]$, say. Let ψ be a unit vector of $\mathbf{H}_a$. The map $\xi_i\otimes\psi\mapsto\xi_i\otimes\gamma_i$ extends to a unitary operator U on $\mathbf{H}\otimes\mathbf{H}_a$ [see, e.g., Theorem 3.2 in Beltrametti *et al.* (1990)], such that $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ is a premeasurement of A. Let $\varphi=\sum v_i\xi_i$ $(v_i\neq 0)$, so that φ is a unit vector of $\mathbf{H}$. Then $\hat{\Phi}=U(\varphi\otimes\psi)$. Clearly, this premeasurement leads also to strong correlations.

The results of this section are summarized in the following corollary.

Corollary. A strong correlation premeasurement of a simple quantity always leads to the polar decomposition of the final vector state of the object–apparatus system. The polar decomposition of any (nondegenerate) vector state of such a compound system can always be interpreted as resulting from a strong correlation premeasurement of some simple quantity.

Note. Theorems 3.2 and 4.2 of Beltrametti *et al.* (1990) show, in fact, that a strong correlation premeasurement of any discrete quantity (simple or not) always leads to the polar decomposition of the final vector state of the object–apparatus system. Moreover, the above justification of the second part of the Corollary can also immediately be repeated for any unit vector (degenerate or not) of $\mathbf{H}\otimes\mathbf{H}_a$.

4. THE POLAR DECOMPOSITION AND THE OBJECTIFICATION PROBLEM

A strong correlation premeasurement $\langle \mathbf{H}_a, A_a, \psi, U\rangle$ of a simple quantity A singles out the polar decomposition of the final state of the object–apparatus system,

$$U(\varphi\otimes\psi)=\sum c_i\varphi_i\otimes\gamma_i \tag{20}$$

and the canonical decompositions of the final states of the object and of the apparatus:

$$\pi_a(P[U(\varphi\otimes\psi)])=\sum |c_i|^2P[\varphi_i] \tag{21}$$

$$\pi(P[U(\varphi\otimes\psi)])=\sum |c_i|^2P[\gamma_i] \tag{22}$$

Assume that an ignorance interpretation of (22) could be given, i.e., when the measuring apparatus $\mathbf{A}$ is in the mixed state $\sum |c_i|^2P[\gamma_i]$ it is, in fact, in one of the pure states $P[\gamma_i]$, $i=1,\ldots,N$, the weights $|c_i|^2$ describing our imperfect knowledge of the actual state of $\mathbf{A}$. This would mean that the pointer observable A_a would have, in fact, a well-defined, though subjectively unknown, value, namely one of the eigenvalues. Due to the strong correlation, this information could then be transferred to the object system as well. In other words, the objectification problem would have been solved. However, the "if" here is, indeed, the applicability of the ignorance interpretation to (22). Though the polar decomposition as well as the canonical decompositions are, in the present case, unique and natural, the decomposition of, e.g., the mixed state $\pi(P[U(\varphi\otimes\psi)])$ into pure components is highly nonunique. In fact, any unit vector $\gamma\in \mathrm{ran}(\{\pi(P[U(\varphi\otimes\psi)])\}^{1/2})$ can appear as a pure component in some decomposition of $\pi(P[U(\varphi\otimes\psi)])$ (Hadjisavvas, 1981). This is the well-known difficulty behind the objectification problem in quantum mechanics. In other words, even though the canonical decomposition of $\pi(P[U(\varphi\otimes\psi)])$ in (22) is unique (as the spectral decomposition) and natural (with respect to the measurement interaction U), it is not the only decomposition of that mixed state into its pure components. The task of the objectification problem is just to find conditions, say on the measuring interaction U and on the measuring apparatus **A**, which would show that, after all, the natural decomposition of $\pi(P[U(\varphi\otimes\psi)])$ given in (22) is its physically relevant decomposition into pure states.

It is now clear from the above discussion that the polar decomposition of an interaction [i.e., of the state $U(\varphi\otimes\psi)$] does not solve the objectification problem. It is an alternative way to explicate the result of a strong correlation premeasurement.

The only systematic solution of the objectification problem within quantum mechanics (known to me) is the following. Assume that the pointer observable A_a is a (nonconstant) classical quantity of **A**, i.e., a quantity which commutes with any other physical quantity of **A**. This assumption then implies, among others, that the only pure states of **A** are the eigenstates of A_a. Then, clearly, $\pi(P[U(\varphi\otimes\psi)])=\sum|c_i|^2P[\gamma_i]$ is the only decomposition of this mixed state into pure states of **A** leading thus to a solution of the objectification problem. This solution has an *ad hoc* character and it also implies some further problems, especially in the connection of the measurement interaction U to the dynamic evolution of the object–apparatus system. However, it serves here to illustrate the objectification problem. Moreover, it shows the consistency of the Born interpretation of the basic probabilities in quantum mechanics. [For further discussion of these points, see, e.g., Beltrametti and Cassinelli (1981), Beltrametti *et al.* (1990), and Mittelstaedt (1976).]

5. CONCLUDING REMARKS

Kochen (1985) proposed the witnessing interpretation of quantum mechanics in order "to show that there is a consistent view of the formalism as describing an objective world of individual interacting systems." I cite Kochen (1985) to identify the core of this new interpretation:

> We now take our major step in the new interpretation. In place of an official human observer, we assume that each system acts as a witness to the state of the other. When the polar decomposition assigns the mixed states $\sum v_i^2P[\gamma_i]$ and $\sum v_i^2P[\xi_i]$ to the two interacting systems **A** and **S**, then **A** actually has exactly one of the properties $P[\gamma_i]$ as witnessed by **S**, and **S** has the corresponding property $P[\xi_i]$ as witnessed by **A**. In other words, if $\hat{\Phi}=\sum v_i\xi_i\otimes\gamma_i$, then **A** and **S** are in one of the corresponding states γ_i and ξ_i. [The citation is exact, though I have adopted here the notations of the present paper.]

The above citation shows clearly that the very heart of the witnessing interpretation is in the claim that when the compound system **S**+**A** is in the vector state $\hat{\Phi} = U(\varphi \otimes \psi) = \sum v_i \xi_i \otimes \gamma_i$, then **S** and **A** are in one of the pure states $P[\xi_i]$ and $P[\gamma_i]$, $i = 1, \ldots, N$. As was demonstrated in Section 4, such a conclusion does not follow from the polar decomposition. This then clearly shows that the witnessing interpretation is, indeed, a radical deviation from or addition to the usual Born interpretation of quantum mechanics.

As a technical remark, note that the above form of the witnessing interpretation clearly assumes that the polar decomposition of $\hat{\Phi}$ is nondegenerate. Whether the nondegeneracy assumption is crucial for the witnessing interpretation remains to be seen. Some recent results (Kochen, 1988*b*) seem to indicate that this is not the case.

Let us consider more closely the witnessing procedure. From Kochen (1985, 1988*a*) it becomes evident (cf. also the above citations) that "strong correlations," which are apparent in the polar decomposition of $\hat{\Phi} = U(\varphi \otimes \psi) = \sum v_i \xi_i \otimes \gamma_i$, play an important role in witnessing. But as already pointed out, this cannot be all of the witnessing procedure. Indeed, for a given degenerate $\hat{\Phi} = U(\varphi \otimes \psi) = \sum v_i \xi_i \otimes \gamma_i$ one may always construct a ξ in $\mathrm{ran}(\pi_a(P[U(\varphi \otimes \psi)])^{1/2})$, $\xi \neq \xi_i$ for all $i = 1, \ldots, N$, and a γ in $\mathrm{ran}(\pi(P[U(\varphi \otimes \psi)])^{1/2})$, $\gamma \neq \gamma_i$ for all $i = 1, \ldots, N$, such that $\rho(P[\xi], P[\gamma], U(\varphi \otimes \psi)) = 1$. Thus, in order that **A**, say, may witness that **S** is in the state $P[\xi_k]$, say, there must be a method for **A** to determine first its own pure state. Indeed, the witnessing interpretation assumes that the states available for **A**, say, are exactly $P[\gamma_1], \ldots, P[\gamma_N]$ and, in addition, that **A** should be in position to decide in which of the possible states it actually is.

The above discussion invites us to compare the witnessing procedure with the old London and Bauer (1983) (the original French text dates from 1929), interpretation of quantum mechanics. This interpretation started with the assumption that quantum mechanics is a universally valid physical theory, applying thus to the observers as well. To solve the objectification problem, London and Bauer went on to assume that the observer can by "introspection" and with his "immanent knowledge" always rightly create his own objectivity, and thus identify his own pure state. With reference to the polar decomposition $\hat{\Phi} = U(\varphi \otimes \psi) = \sum v_i \xi_i \otimes \gamma_i$, the "observer" **A**, indeed, would, according to London and Bauer, have the right to say that "I am in the state $P[\gamma_k]$"! The citations from Kochen (1985) indicate that the witnessing interpretation should not be identified with the London–Bauer interpretation.

Neither in Kochen (1985) nor in Kochen (1988*a*) are the conceptual foundations of the witnessing interpretation of quantum mechanics worked out in a systematic way. Hence, it is outside the scope of the present paper to study the consistency of that interpretation or to try to evaluate its relation to the Born interpretation. A careful conceptual reconstruction of the witnessing/perspective interpretation of quantum mechanics is important, and the more so since in Kochen (1988*a*) it is shown that, in fact, the witnessing interpretation leads to some predictions which are beyond the usual theory. In particular, it was shown in Kochen (1988*a*) that the witnessing interpretation of quantum mechanics implies that the only statistics available for systems of identical particles are the Bose and the Fermi statistics.

In a series of papers Goernitz and von Weizsäcker (1987*a*,*b*) studied different interpretations of quantum mechanics. In particular, in studying the witnessing interpretation of quantum mechanics they came to the conclusion that this interpretation "is essentially identical with the Copenhagen interpretation." As was shown in the present paper, the polar decomposition of an interaction has, indeed, a measurement-theoretic interpretation in terms of a strong correlation premeasurement. On the basis of this paper the claim of the essential identification of the Copenhagen interpretation, in any of its historical forms, and of the witnessing interpretation appears, however, too strong.

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